Reflection and refraction of a plane wave at oblique incidence

Let us consider a plane wave that obliquely incidents at the boundary of two media that are characterized by their permittivity and permeability (see Figure 1). The plane containing both the normal to the surface and the direction of propagation of the incident wave is known as the plane of incidence.

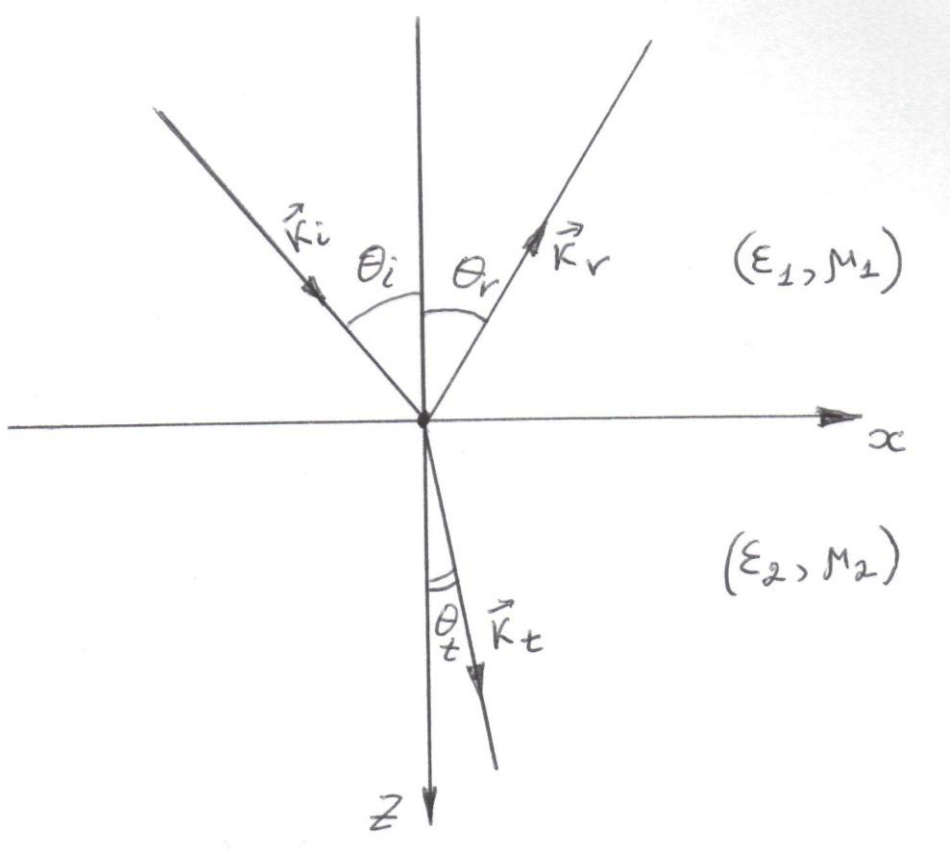

Fig. 1

We consider two different cases:

Case **A**: The electric field of the incident wave is perpendicular to the plane of incidence.

Case **B**: The electric field of the incident wave is in the plane of incidence.

Any other incident wave can be decomposed into linear combination of these two.

Fig. 1 shows a wave of either polarization incident on the boundary of two media. In this Figure, the angle θ_i between the normal to the boundary and the propagation direction is the angle of incidence. We choose the plane of incidence to be the $x-z$ plane with the axes directions shown in the Figure1, the y-axes is out of the page.

There may also be reflected and refracted (transmitted) waves, as shown in Fig. 1. Directions of propagation of these waves have angles θ_r and θ_t with the normal to the boundary. The unit wave vectors of the incident, reflected and transmitted waves can be written as the following:

$$\hat{k}_i = [\sin\theta_i, 0, \cos\theta_i] \tag{1}$$

$$\hat{k}_r = [\sin\theta_r, 0, -\cos\theta_r] \tag{2}$$

$$\hat{k}_t = [\sin\theta_t, 0, \cos\theta_t] \tag{3}$$

The phasors of the traveling incident, reflected, and refracted plane waves can be written in the following form

$\vec{E}_i(\vec{r}) = E_0 \exp(-i\vec{k}_i \cdot \vec{r})$, $\vec{E}_r(\vec{r}) = E_1 \exp(-i\vec{k}_r \cdot \vec{r})$, and $\vec{E}_t(\vec{r}) = E_2 \exp(-i\vec{k}_t \cdot \vec{r})$

correspondingly. If the wave fields depend on coordinate as $\exp(-i\vec{k} \cdot \vec{r})$, then Maxwell equations for phasors

$$\vec{\nabla} \times \vec{E} = -i\omega\mu\vec{H} \tag{4}$$

$$\vec{\nabla} \times \vec{H} = i\omega\varepsilon\vec{E}$$

can be written in the form:

$$-i(\vec{k} \times \vec{E}) = -i\omega\mu\vec{H} \tag{4'}$$

$$-i(\vec{k} \times \vec{H}) = i\omega\varepsilon\vec{E} \tag{5'}$$

Now we can write the wave fields for the cases **A** and **B**,

Incident wave, $\vec{E}_i, \vec{H}_i \sim \exp\{-ik_1(x\sin\theta_i + z\cos\theta_i)\}$

Case **A** (E perpendicular to plane of incidence)

$$\begin{aligned} \vec{E}_i &= \hat{j}E_0 \exp\{-i\vec{k}_i\vec{r}\} \\ \vec{H}_i &= E_0\sqrt{\frac{\varepsilon_1}{\mu_1}}(-\hat{i}\cos\theta_i + \hat{k}\sin\theta_i)\exp\{-i\vec{k}_i\vec{r}\} \end{aligned} \tag{6}$$

Case **B (** E in plane of incidence)

$$\vec{E}_i = E_0(\hat{i}\cos\theta_i - \hat{k}\sin\theta_i)\exp\{-i\vec{k}_i\vec{r}\}$$
$$\vec{H}_i = \hat{j}E_0\sqrt{\frac{\varepsilon_1}{\mu_1}}\exp\{-i\vec{k}_i\vec{r}\} \tag{7}$$

Reflected wave, $\vec{E}_r, \vec{H}_r \sim \exp\{-ik_1(x\sin\theta_r - z\cos\theta_r)\}$

Case **A (** E perpendicular to plane of incidence)

$$\vec{E}_r = \hat{j}E_1\exp\{-i\vec{k}_r\vec{r}\}$$
$$\vec{H}_r = E_1\sqrt{\frac{\varepsilon_1}{\mu_1}}(\hat{i}\cos\theta_r + \hat{k}\sin\theta_r)\exp\{-i\vec{k}_r\vec{r}\} \tag{8}$$

Case **B (** E in plane of incidence)

$$\vec{E}_r = E_1(\hat{i}\cos\theta_r + \hat{k}\sin\theta_r)\exp\{-i\vec{k}_r\vec{r}\}$$
$$\vec{H}_r = -\hat{j}E_1\sqrt{\frac{\varepsilon_1}{\mu_1}}\exp\{-i\vec{k}_r\vec{r}\} \tag{10}$$

Refracted (transmitted) wave, $\vec{E}_t, \vec{H}_t \sim \exp\{-ik_2(x\sin\theta_t + z\cos\theta_t)\}$

Case **A (** E perpendicular to plane of incidence)

$$\vec{E}_t = \hat{j}E_2\exp\{-i\vec{k}_{ti}\vec{r}\}$$
$$\vec{H}_t = E_2\sqrt{\frac{\varepsilon_2}{\mu_2}}(-\hat{i}\cos\theta_t + \hat{k}\sin\theta_t)\exp\{-i\vec{k}_t\vec{r}\} \tag{11}$$

Case **B (** E in plane of incidence)

$$\vec{E}_t = E_2(\hat{i}\cos\theta_t - \hat{k}\sin\theta_t)\exp\{-i\vec{k}_t\vec{r}\}$$
$$\vec{H}_t = \hat{j}E_0\sqrt{\frac{\varepsilon_2}{\mu_2}}\exp\{-i\vec{k}_t\vec{r}\} \tag{12}$$

We introduced unit vectors $\hat{i}, \hat{j}, \hat{k}$ along x, y, z axes in Eqs. (6)-(12).

Case B: Let us consider a plane wave with electric field $\vec{E}$ in the plane of incidence incident on the discontinuity between two dielectrics $(\varepsilon_1, \mu_1),(\varepsilon_2, \mu_2)$

The boundary conditions at $z = 0$ are continuity of tangential components of electric and magnetic fields $\vec{E}_{\tan}$ and $\vec{H}_{\tan}$:

$$\begin{aligned} &E_0 \cos\theta_i \exp(-ik_1 x \sin\theta_i) + E_1 \cos\theta_r \exp(-ik_1 x \sin\theta_r) = E_2 \cos\theta_t \exp(-ik_2 x \sin\theta_t) \\ &\sqrt{\frac{\varepsilon_1}{\mu_1}} E_0 \exp(-ik_1 x \sin\theta_i) - \sqrt{\frac{\varepsilon_1}{\mu_1}} E_1 \exp(-ik_1 x \sin\theta_r) = \sqrt{\frac{\varepsilon_2}{\mu_2}} E_2 \exp(-ik_2 x \sin\theta_t) \end{aligned} \tag{13}$$

Eqs. (13) must hold for all values of x, which is possible only if

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \tag{14}$$

We see that

$\theta_i = \theta_r$ (angle of refraction equals angle of incidence) (15)

and

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2} = \frac{\omega\sqrt{\varepsilon_1\mu_1}}{\omega\sqrt{\varepsilon_2\mu_2}} = \frac{v_{ph}^{(2)}}{v_{ph}^{(1)}} = \frac{n_1}{n_2} \tag{16}$$

Here n is index of refraction.

Eq. (16) is the familiar Snell's law.

Canceling the exponential terms in (13), (14) by means of (15), we obtain

$$\begin{aligned} &E_0 \cos\theta_i + E_1 \cos\theta_i = E_2 \cos\theta_t \\ &\sqrt{\frac{\varepsilon_1}{\mu_1}} E_0 - \sqrt{\frac{\varepsilon_1}{\mu_1}} E_1 = \sqrt{\frac{\varepsilon_2}{\mu_2}} E_2 \end{aligned} \tag{17}$$

We can now solve Eqs. (17) for E_1 and E_2 with the result

$$\begin{aligned}
\frac{E_1}{E_0} &= \frac{\sqrt{\varepsilon_1/\mu_1}\cos\theta_t - \sqrt{\varepsilon_2/\mu_2}\cos\theta_i}{\sqrt{\varepsilon_1/\mu_1}\cos\theta_t + \sqrt{\varepsilon_2/\mu_2}\cos\theta_i} \\
\frac{E_2}{E_0} &= \frac{2\sqrt{\varepsilon_1/\mu_1}\cos\theta_i}{\sqrt{\varepsilon_1/\mu_1}\cos\theta_t + \sqrt{\varepsilon_2/\mu_2}\cos\theta_i}
\end{aligned} \tag{18}$$